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# THE BEHAVIOR OF A WING PANEL IN A STREAM OF GAS UNDER TRANSIENT CONDITIONS

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A method is proposed for investigating the behavior of an elastic flat panel of aircraft wing skin in a stream of gas under transient conditions. The panel is assumed to have an initial deflection. The solution is based on the wave equation of linearized unsteady aerodynamics and the geometrically nonlinear equations of the theory of elastic plates [1]. The resolving equations which define the behavior of an elastic system are derived with the use of the Bubnov-Galerkin procedure with respect to one of the coordinates and of the method of finite differences with respect to the other coordinate and time. As an example, the supersonic flow past a wing is considered. Aerodynamic pressure distribution over the panel surface is determined, using a thin lift surface as the model, by the numerical method of retarded source potential, taking into consideration previous history of the deformation process [2].

Let a wing of rectangular plan form, moving in the direction of its axis of symmetry at velocity $U_0$ and zero angle of attack in a perfect compressible fluid, be subjected at instant $t = 0$ to small additional transient motions caused by instantaneous variation of the angle of attack $\alpha$ induced by vertical gusts. It is assumed that in the following instants individual sections of the wing skin begin to distort, and that the flow around the wing is streamlined.

Let us analyze the dynamic reaction of an elastic system subjected to a sudden change of stream parameters on the example of a flat wing skin panel section of sides $a$ and $b$ and thickness $h$ (Fig. 1). Let us assume that this plate is attached by hinges to the structure reinforcing members, is loaded in its plane by compressive stresses $p$, has an initial deflection, and lies in the downstream Mach cone. Note that in this case the end effects and the vortex sheet do not affect pressure distribution over the panel surface. The direction of flow and of positive axes of the wing $O_1x_1y_1$ and of the panel $Oxy$ are shown in Fig. 1. The $z_1$-coordinate is normal to the wing plane and $z = -z_1$, $y = -y_1$ is assumed for panel coordinates.

Let us write down the system of differential equations which define the distribution of the perturbed velocity potential $\varphi$ in the moving system of coordinates $O_1x_1y_1$ and the considerable deflections of the panel, with allowance for structural damping and inertia forces, which correspond to normal deflection

$$(a_0^2 - U_0^2)\varphi_{,x_1x_1} + a_0^2\varphi_{,y_1y_1} + a_0^2\varphi_{,z_1z_1} - \varphi_{,tt} - 2U_0\varphi_{,x_1t} = 0 \tag{1}$$

$$\frac{D}{h}\nabla^2\nabla^2(w - w_0) = L(w, \Phi) - \varepsilon w_{,t} - \frac{\gamma}{g} w_{,tt} + \frac{p_a(x, y, t)}{h} \tag{2}$$

$$\frac{1}{E}\nabla^2\nabla^2\Phi = -{}^1/_2\,[L(w, w) - L(w_0, w_0)] \tag{3}$$

In these formulas $a_0$ is the speed of sound in the unperturbed stream, $w$ and $w_0$ are, respectively, the total and initial deflections, $\Phi$ is a function of stresses in the median surface, $D$ is the cylindrical rigidity, $\gamma$ is the specific weight of the plate material, $\varepsilon$ is the coefficient of structural damping, E is the modulus of elasticity of the material, $p_a(x, y, t)$ is the intensity of the transverse aerodynamic load, and $L$ is a known bilinear operator.

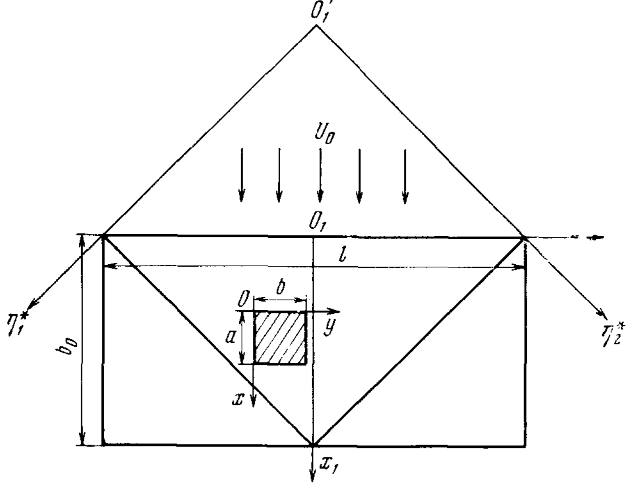


Fig. 1

The boundary conditions for the panel are

$$[\varphi_{,z_1}]_{z_1=0} = \begin{cases} 0, & t < 0 \\ U_0\left(\alpha + w_{,x} + \frac{1}{U_0} w_{,t}\right), & t \geqslant 0 \end{cases} \tag{4}$$

Parameters $w_{,x}$ and $w_{,t}$ represent the variation of deflection of the panel median plane with respect to the $x$-coordinate and time $t$, respectively.

We introduce the dimensionless parameters

$$\zeta = \frac{f}{h}, \quad \zeta_0 = \frac{f_0}{h}, \quad \xi = \frac{x}{a}, \quad \lambda = \frac{a}{b}, \quad \psi = \frac{\Phi_0}{Eh^2}, \quad p^* = \frac{pb^2}{Eh^2}$$

$$p_a^* = \frac{2(p_a - p_\infty)}{\rho_\infty U_0^2}, \quad \varepsilon^* = \frac{\varepsilon a^2}{h}\sqrt{\frac{g}{E\gamma}}, \quad \vartheta = \sqrt{12(1-\nu^2)}, \quad \lambda_1 = \frac{a}{h} \tag{5}$$

$$\lambda_2 = \frac{a}{b_0}, \quad \rho^* = \frac{\rho_\infty}{\rho_0}, \quad V^* = \frac{V}{a_0}, \quad a_1 = \frac{V^*}{\lambda_1\lambda_2 M}, \quad a_2 = \frac{2\rho^*}{\pi}\left(\frac{\lambda_1}{\lambda_2}\right)^2, \quad \tau = \frac{U_0 t}{b_0}$$

where $p_\infty$ and $\rho_\infty$ are, respectively, the pressure and the density of the medium in the unperturbed stream, $\rho_0$ is the density and $\nu$ the Poisson's ratio of the plate material, $V$ is the speed of sound in the plate material, and $b_0$ is the wing chord.

We approximate functions of total and initial deflections and the function of stresses in the median plane by expressions of the form

$$w = f(x, t)\sin\frac{n\pi y}{b}, \quad w_0 = f_0(x)\sin\frac{n_0\pi y}{b}, \quad \Phi = \Phi_0(x, t)\sin\frac{n\pi y}{b} - \frac{py^2}{2} \tag{6}$$

where $n$ and $n_0$ are, respectively, the numbers of specified current and initial half-waves along the deflection arc. We set $n = n_0$.

Substituting formulas (6) into relationships (2) and (3) and applying the Bubnov-Galerkin procedure with respect to the $y$-coordinate with allowance for (5), we obtain

$$\begin{gathered}\zeta_{,\tau\tau} + \varepsilon^* a_1 \zeta_{,\tau} = a_2 p_a^*(\xi, \tau) - a_3 [(\zeta - \zeta_0)_{,\xi\xi\xi\xi} - 2(n_0\pi\lambda)^2(\zeta - \zeta_0)_{,\xi\xi} + \\ (n_0\pi\lambda)^4(\zeta - \zeta_0) + {}^8/_3\vartheta\pi\lambda^2(\psi\zeta_{,\xi\xi} + \psi_{,\xi}\zeta_{,\xi} + \zeta\psi_{,\xi\xi}) + p^*\vartheta\lambda^2\zeta_{,\xi\xi}]\end{gathered} \tag{7}$$

$$\begin{gathered}\psi_{,\xi\xi\xi\xi} - 2(n_0\pi\lambda)^2\psi_{,\xi\xi} + (n_0\pi\lambda)^4\psi = {}^8/_3 n_0\pi\lambda^2 \times \\ \left[\zeta\zeta_{,\xi\xi} + {}^1/_2\zeta_{,\xi}^2 - \zeta_0\zeta_{0,\xi\xi} - {}^1/_2\zeta_{0,\xi\xi}^2\right], \quad a_3 = \frac{a_1^2}{\vartheta}\end{gathered} \tag{8}$$

The combined solution of Eqs. (7) and (8) and the determination of the pressure parameter $p_a^*(\zeta, \tau)$ by numerical methods make it possible to observe the deformation of the panel with time in transient conditions of flow for a wide range of Mach numbers $0 \leqslant M \leqslant 2$.

As an example, let us analyze the laws of aerodynamic load distribution and panel deformation in the case of supersonic flow of gas past the wing at $M = \sqrt{2}$.

In accordance with (4) and (5) we represent $p_a^*(\xi, \tau)$ in the form

$$p_{a_i}^*(\xi, \tau) = p_{a_i}^*(\xi, \tau) + p_{\zeta i,\xi}^*(\xi, \tau) + p_{\zeta i,\tau}^*(\xi, \tau), \quad i = 1, 2, 3 \ldots \tag{9}$$

$$\begin{gathered}[p_{\zeta i,\xi}^*(\xi, \tau) + p_{\zeta i,\tau}^*(\xi, \tau)] = \frac{2l}{b_0}\left[\frac{\varphi^*(\tau + h_{\tau_a}) - \varphi^*(\tau)}{h_{\tau_a}} + \frac{\varphi^*(\xi_1 + h_{\xi_1}) - \varphi^*(\xi_1 - h_{\xi_1})}{2h_{\xi_1}}\right] \\ h_{\tau_a} = 2/N_1, \quad h_{\xi_1} = 1/N_1, \quad \xi_1 = x_1/b_0, \quad \varphi^* = 2\varphi/U_0 l\end{gathered} \tag{10}$$

where $p_{a_i}^*(\xi, \tau)$ is the dimensionless component of aerodynamic pressure at any point of the panel, which in this case is determined by simple formulas (see [2], p. 547), $l$ is the wing span, $h_{\tau_a}$ and $h_{\xi_1}$ are the steps of aerodynamic pressure integration with respect to time and with respect to the wing chord, respectively, $N_1$ is the number of chord subdivisions.

To integrate Eqs. (1) by the numerical method of retarded source potential we introduce dimensionless characteristic coordinates $0_1'\eta_1^*\eta_2^*$ with the vertex half-angle

$\mu = \arcsin(M^{-1})$ (Fig. 1), defined by

$$\eta_1{}^* = x_1{}^* - y_1{}^* - x_{10}{}^*, \quad \eta_2{}^* = x_1{}^* + y_1{}^* - x_{10}{}^*, \quad \beta = \sqrt{M^2 - 1}$$

$$y_1{}^* = \frac{2y_1}{l}, \quad x_1{}^* = \frac{2x_1}{\beta l}, \quad x_{10}{}^* = \frac{2x_{10}}{\beta l} \tag{11}$$

where $x_{10}$ is the reference coordinate.

In order to apply the numerical method we divide the panel along its side $a$ into $N$ equal parts and draw through these points straight lines parallel to the characteristic lines. In this way the actual panel is replaced by an integral number of elementary surfaces. We determine the perturbed velocity potential $\varphi_i{}^*$ at any point by summating integrals over each surface element belonging to the integration region

$$\varphi_i{}^*(F\Delta, H\Delta, \tau) = \frac{\Delta}{\pi}\sum_{i=1}^{F}\sum_{j=1}^{H} C_i(\tau_1)\, r_i{}^* + \frac{\Delta}{\pi}\sum_{i=1}^{F}\sum_{j=1}^{H} C_i(\tau_2)\, r_i{}^* \tag{12}$$

$$C_i(\tau) = [\zeta_{i,\xi} + \zeta_{i,\tau}] \sin\pi\left(\frac{\Delta}{2}\beta_i\right)$$

$$F = \eta_1{}^*/\Delta, \qquad H = \eta_2{}^*/\Delta$$

$$r_i{}^* = (\sqrt{F - i + 1} - \sqrt{F - i})(\sqrt{H - j + 1} - \sqrt{H - j})$$

$$\tau_{1,2} = \tau - \frac{M^2 l\Delta}{2\beta b_0}\left[\frac{F - i + H - j}{2} \mp \frac{\sqrt{(F-i)(H-j)}}{M}\right]$$

where $\Delta$ is the length of the surface element side, and $\zeta_{i,\xi}$ and $\zeta_{i,\tau}$ reflect local variations of panel deformation with respect to the $\xi$-coordinate and time $\tau$, respectively, and are determined by the solution of Eqs. (7) and (8). Parameter $\Delta\beta_i / 2$ defines the position of a surface element along side $b$ of the panel (Fig. 2).

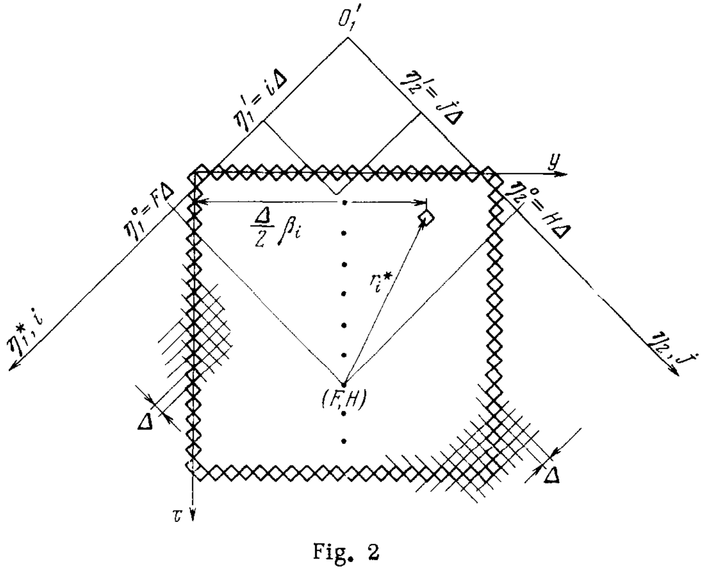


Fig. 2

Note that the signal from a surface element belonging to the integration area may

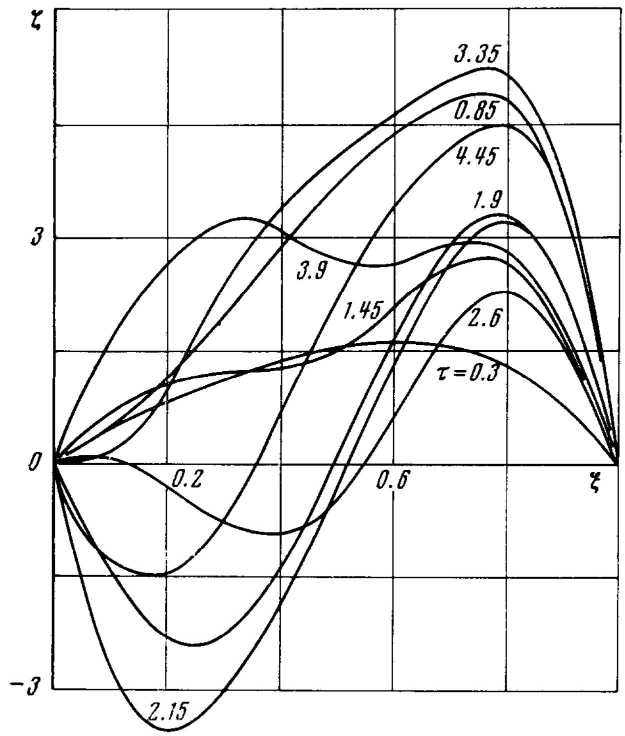
ζ
3.35
0.85
4.45
1.9
3
3.9
1.45
2.6
τ = 0.3
0
0.2
0.6
ζ
−3
2.15

Fig. 3

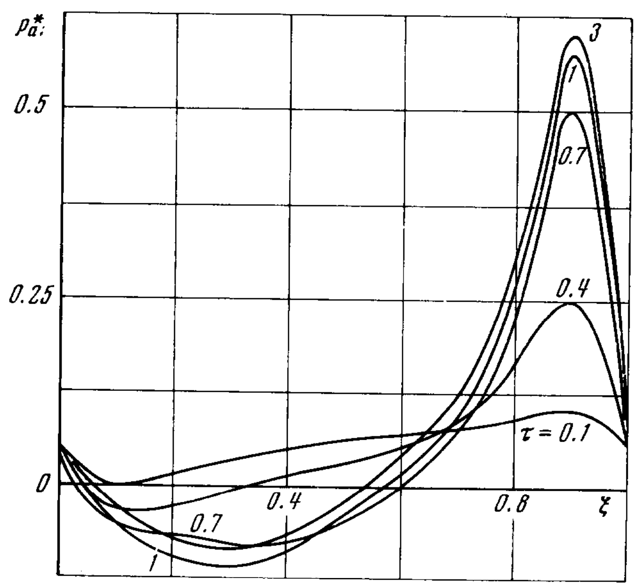
$p_a^*$;
3
1
0.5
0.7
0.4
0.25
τ = 0.1
0
0.4
0.8
ζ
0.7
1

Fig. 4

reach an investigated point at instant $\tau$, either from the forward wave front ($\tau_1 > 0$) or from the rear one ($\tau_2 > 0$), or simultaneously from the two fronts.

Equations (7) and (8) were integrated on a computer over finite differences with steps $h_\xi = 0.1$ with respect to the coordinate and $h_\tau = 0.005$ with respect to time, for initial conditions $\zeta - \zeta_0 = \zeta_{,\tau} = 0$ together with (12), (10) and (9) for $h_{\xi_1} = 0.0125$ and $h_{\tau_a} = 0.025$. The derivatives of functions $\zeta$ and $\psi$ with respect to the $\xi$-coordinate and time $\tau$ were replaced at node $(i, j)$ by central-difference operators. A Dural panel with parameters $\lambda_1 = 200$, $\lambda_2 = 0{,}25$, $n_0 = 1$, $\lambda = 1$, $\zeta_0 = 1 \cdot \sin \pi\xi$, and $\varepsilon = 0{,}25 \cdot 10^{-3}$ kg. sec/cm$^4$ was taken as the example.

The calculation begins by the determination of $p_{a_i}{}^* = p_{\alpha_i}{}^*$ for $\tau = 0$. Then, setting $p_{a_i}{}^* = \text{const}$, the process of determination of $\zeta_i{}^j$ and $\psi_i{}^j$ by (7) and (8) is repeated until the cumulative value of $h_\tau$ becomes equal $h_{\tau_a}$. After this we determine parameter $(\zeta^j_{i,\xi} + \zeta^j_{i,\tau})$, using the obtained values of $\zeta_i{}^j$ for the $j$th layer and of $\zeta_i{}^0$ for the zero layer at points lying along line $y = b/2$ and retain this parameter at each step $h_{\tau_a}$.

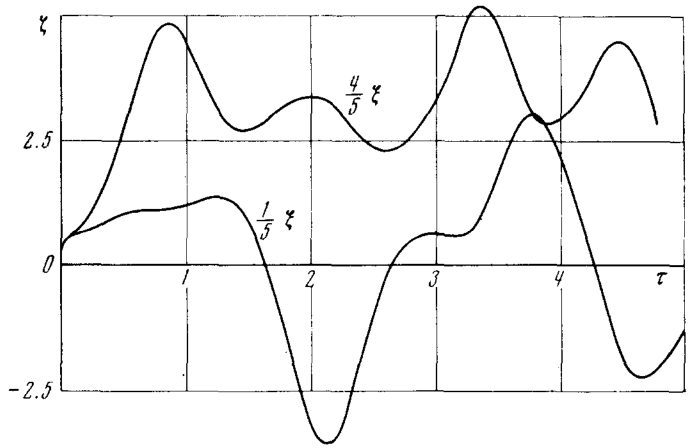


Fig. 5

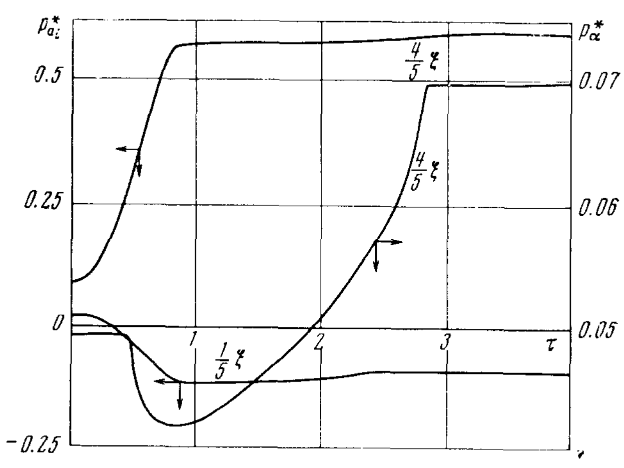


Fig. 6

Then, using (9) with allowance for (12) and (10), we determine $p_{a_i}{}^*$ at the same points. This is followed by the determination of functions $\zeta_i{}^j$ and $\psi_i{}^j$ for the next layer, using the new value of $p_{a_i}{}^*$. This procedure is repeated until the criterion of completion of the transient process – stabilization of the oscillation mode of the elastic system – is reached (Fig. 5). Completion of this process can be also judged by the behavior of transient functions $p_{a_i}{}^*(\tau)$ (see Fig. 6).

The obtained numerical data are shown in Figs. 3 and 4 in the form of curves $\zeta(\xi, \tau)$ and $p_a{}^*(\xi, \tau)$ for $p^* = 3$. Examination of Fig. 3 shows that the continuous change of the median surface shape with a shift of the maximum amplitude of deflection toward the plate trailing edge is a distinctive feature of deformation of the panel under transient conditions. This also follows from the diagram of distribution of parameter $p_a{}^*(\xi, \tau)$ shown in Fig. 4. A set of curves related to the points lying at distances equal $^1/_5$ and $^4/_5$ of the plate length from the plate leading edge is shown in Fig. 5. It is seen that perturbations in the stream result in rapid increase of deflection, beginning at points in the neighborhood of the trailing edge and, then, with a certain lag, at points close to the leading edge. Later, when oscillations become stabilized, this process is somewhat attenuated. Transient functions $p_{\alpha_i}{}^*(\tau)$ and $p_{ai}{}^*(\tau)$ for points of the panel to which in Fig. 5 relate curves $\zeta_i(\tau)$ are shown in Fig. 6, where the different rates of pressure increase with time are a significant feature of curves $p_{a_i}{}^*(\tau)$.

We note in conclusion that the proposed method makes it possible not only to establish the deformation pattern of the panel median plane and pressure distribution with respect to time but, also, to determine unsafe stresses in the structure in transient mode and stable oscillations.

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# PLANE CONTACT PROBLEM FOR A LINEARLY-DEFORMABLE FOUNDATION IN THE PRESENCE OF ADHESION

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On the basis of a spectral relationship for the Jacobi polynomials which is more general than that used in [1], a method is proposed for solving the plane contact problem for a linearly-deformable foundation of general type whose particular case is a half-space with an elastic modulus of the form

$$E = E_\nu z^\nu \qquad (0 \leqslant \nu < 1)$$

The exact solution of the plane problem of the impression of a die with